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The Stern–Gerlach quantum-like behaviour of a classical charged particle

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Abstract. The effect of the radiation reaction on the precession of a classical charged particle in a magnetic field is investigated. It is shown that the spin spirals around the field and becomes parallel or antiparallel to it, depending on the initial value of the angle between spin and field being smaller or greater than 90°.

1. Introduction

The space quantisation of angular momentum is one of the most characteristic features of quantum mechanics. The Stern–Gerlach experiments and the Zeeman effect which show this quantization are usually considered as two very sound tests of the basic ideas of quantum theory. In some textbooks (Feynman 1965) the Stern–Gerlach experiment is presented as the archetype of the quantum process of measure and it is assumed that it cannot be explained on classical grounds. On the other hand, there have been some proposals of classical deterministic models to account for the results of the Stern–Gerlach experiments, the most interesting being the one by Bohm (Bohm and Bub 1966). However, they are not generally considered as valid realistic models.

There is however a point to be noted. The classical treatment of the motion of spins, which is usually presented in quantum mechanics books, is based on the idea of the Larmor precession and on a linear evolution equation. Its phase space is a sphere with two centres in the poles, corresponding to parallel or antiparallel spin. But a well known theorem by Liapunov, the theorem on the stability in the linear approximation (Meirovitch 1970), asserts that centres in linear equations correspond to a critical case and may change into nodes, saddles or spiral points if an arbitrarily small nonlinear perturbation is added. This implies that any new effect, however small, could qualitatively change the situation. If the centres become spiral attractors the spin would align or anti-align with the magnetic field in a deterministic continuous way. Nevertheless spiral points cannot arise in conservative systems, but they are typical of dissipative systems. This suggests the study of the emission of radiation which through its reaction on the particle could change qualitatively the nature of the equilibrium points at the poles.

We will see that this is the case for a charged particle. The kind of particle that we have in mind is an extended system, a kink or soliton of a nonlinear field equation. Some models of this type have been proposed previously (Rañada *et al* 1974, Rañada and Vázquez 1976).

In § 2 we study the correction that the reaction imposes on the Larmor precession. In § 3 we show that these corrections produce an alignment or anti-alignment of the spin of a charged particle changing the poles of the phase space into attractor spiral points. In § 4 we summarise the results and state some conclusions.

2. Radiative corrections to the Larmor precession

Let us consider a classical extended particle, with charge e and magnetic moment $\boldsymbol{\mu}$, subjected to a magnetic field $\boldsymbol{B} = (0, 0, B)$ directed along the Z axis. The torque on the spin of the particle produces a variation in time of $\boldsymbol{\mu}$ in such a way that in the radiation zone the electric and magnetic field take the values (Jackson 1975)

$$\boldsymbol{E} = \frac{e\boldsymbol{n}}{4\pi r^2} + \frac{\boldsymbol{n} \times \dot{\boldsymbol{\mu}}_{\mathrm{R}}}{4\pi c r^2} + \frac{\boldsymbol{n} \times \ddot{\boldsymbol{\mu}}_{\mathrm{R}}}{4\pi c^2 r} \tag{1}$$

$$\boldsymbol{B} = \frac{3\boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{\mu}_{\mathrm{R}}) - \boldsymbol{\mu}_{\mathrm{R}}}{4\pi r^3} + \frac{3\boldsymbol{n}(\boldsymbol{n} \cdot \dot{\boldsymbol{\mu}}_{\mathrm{R}}) - \dot{\boldsymbol{\mu}}_{\mathrm{R}}}{4\pi c r^2} + \frac{\boldsymbol{n} \times (\boldsymbol{n} \times \ddot{\boldsymbol{\mu}}_{\mathrm{R}})}{4\pi c^2 r} \tag{2}$$

where r is the vector from the particle to the observation point, $\boldsymbol{n} = \boldsymbol{r}/r$ and the subscript R indicates that the corresponding quantities must be taken at the retarded time $t_{\mathrm{R}} = t - r/c$. The terms in $1/r$ correspond to a magnetic dipole radiation.

Let us calculate the flux of angular momentum through a sphere of radius r in the limit when $r \to \infty$. We must use the angular momentum density tensor

$$\mathbf{M}^{\alpha\beta\gamma} = \Theta^{\alpha\beta} x^{\gamma} - \Theta^{\alpha\gamma} x^{\beta} \tag{3}$$

where $\Theta^{\alpha\beta}$ is the symmetric energy–momentum tensor

$$\Theta^{\alpha\beta} = F^{\alpha}_{\lambda} F^{\lambda\beta} + \tfrac{1}{4} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu}. \tag{4}$$

The flux of the component J_i of angular momentum through the sphere of radius r takes the value

$$O_i = \int_{4\pi} \mathbf{M}^{jkl} \, \mathrm{d}S_j \tag{5}$$

where $\mathrm{d}S_j$ is the surface element and (j, k, l) is an even permutation of $(1, 2, 3)$. From equations (3)–(5) it is easy to deduce that

$$\boldsymbol{O} = -r^3 \int_{4\pi} [(\boldsymbol{E} \cdot \boldsymbol{n})(\boldsymbol{E} \times \boldsymbol{n}) + (\boldsymbol{B} \cdot \boldsymbol{n})(\boldsymbol{B} \times \boldsymbol{n})] \, \mathrm{d}\Omega \tag{6}$$

and from equations (1), (2) and (6) we obtain

$$\boldsymbol{O} = -(2e/3c^2)\ddot{\boldsymbol{\mu}}_{\mathrm{R}} - (2/3c^3)(\dot{\boldsymbol{\mu}}_{\mathrm{R}} \times \ddot{\boldsymbol{\mu}}_{\mathrm{R}}). \tag{7}$$

The effect of this escape of angular momentum can be represented by a reaction torque $-\boldsymbol{O}$ in the equation of evolution of the spin, which takes the form

$$\dot{\boldsymbol{S}} = \boldsymbol{\mu} \times \boldsymbol{B} + (2e/3c^2)\ddot{\boldsymbol{\mu}} + (2/3c^3)(\dot{\boldsymbol{\mu}} \times \ddot{\boldsymbol{\mu}}) \tag{8}$$

where we have omitted the subscript R because in equation (8) all terms must be taken at the same time. The second and third terms in the right-hand side represent radiative corrections to the Larmor precession. In the case of macroscopic distributions of electric current $\boldsymbol{\mu} = \gamma \boldsymbol{S}$, $\gamma = e/2mc$; while for spinor systems we have $\boldsymbol{\mu} = g\gamma\boldsymbol{S}$, g being 2 if there is no anomalous magnetic moment. It must be stressed that this relation is compatible with a classical treatment of the Dirac field, as shown elsewhere (Rañada *et al* 1974, Rañada and Vázquez 1976). It can also be obtained with a classical treatment of the solutions of the Dirac equation in the hydrogen atom. Using $g = 2$, equation (8) takes the form

$$\dot{\boldsymbol{S}} = (e/mc)(\boldsymbol{S}\times\boldsymbol{B}) + (2e^2/3mc^3)\ddot{\boldsymbol{S}} + (2e^2/3m^2c^5)(\dot{\boldsymbol{S}}\times\ddot{\boldsymbol{S}}). \tag{9}$$

In the case of electrons and other elementary particles the third term is much smaller than the second one and can be neglected, thus obtaining the equation

$$\dot{\boldsymbol{S}} = (e/mc)(\boldsymbol{S}\times\boldsymbol{B}) + (2e^2/3mc^3)\ddot{\boldsymbol{S}} \tag{10}$$

which has a strong similarity with the Abraham–Lorentz equation (Jackson 1975)

$$\dot{\boldsymbol{v}} = (\boldsymbol{F}/m) + (2e^2/3mc^3)\ddot{\boldsymbol{v}}. \tag{11}$$

3. Evolution of the spin in a magnetic field

Let us study equation (10) and, more specifically, the effect of the $\ddot{\boldsymbol{S}}$ term. First of all, this term has a component which is parallel to $\boldsymbol{S}$ and which tends to modify its absolute value. We make the assumption that the internal structure of the particle reacts to the modification of the modulus of its spin. An alternative view could be that a parallel component produces a rotation of the particle on its own axis. This rotation however causes a modification of the angular momentum which is negligible for the magnetic fields attainable in the laboratories. In any case, we will concentrate on the evolution of the direction of the spin. For this reason, we will only consider the tangential component of $\ddot{\boldsymbol{S}}$, which is $[\ddot{\boldsymbol{S}} - (\ddot{\boldsymbol{S}}\cdot\boldsymbol{\nu})\boldsymbol{\nu}]$, where $\boldsymbol{\nu} = \boldsymbol{S}/S$.

Instead of equation (10) we consider thus

$$\dot{\boldsymbol{S}} = (e/mc)(\boldsymbol{S}\times\boldsymbol{B}) + \tau[\ddot{\boldsymbol{S}} - (\ddot{\boldsymbol{S}}\cdot\boldsymbol{\nu})\boldsymbol{\nu}] \tag{12}$$

where $\tau = 2e^2/3mc^3 = 6{\cdot}3\times10^{-24}$ s.

If α and β are the azimuthal and polar angles of the spin we have

$$\boldsymbol{S} = S(\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta)$$

which after substitution in equation (12) gives

$$\begin{aligned} \dot{\alpha} &= \omega + \tau(\ddot{\alpha} + 2\dot{\alpha}\dot{\beta}\cot\beta) \\ \dot{\beta} &= \tau(\ddot{\beta} - \dot{\alpha}^2\sin\beta\cos\beta) \end{aligned} \tag{13}$$

ω being the Larmor frequency. These equations have one of the problems of the Abraham–Lorentz or Lorentz–Dirac equations: the existence of runaway solutions. Two of them are the following:

$$\alpha = \omega t + \exp(t/\tau), \qquad \beta = \pi/2$$

and, if $B = 0$,

$$\alpha = \alpha_0, \qquad \beta = \exp(t/\tau).$$

In order to avoid this problem and to find the right solutions, we follow the standard procedure of dealing with the Lorentz–Dirac or Abraham–Lorentz equations (Jackson 1975, Rohrlich 1965, Barut 1964). Using the identity

$$\ddot{x} - \tau \dot{x} = \tau \exp(t/\tau)(\mathrm{d}/\mathrm{d}t)[\dot{x} \exp(-t/\tau)]$$

and stating that $\dot{\alpha}$ and $\dot{\beta}$ must be bounded when $t \to \infty$, equations (13) can be written as the two coupled integro–differential equations

$$\begin{aligned} \dot{\alpha}(t) &= \omega + \int_0^\infty \exp(-\sigma/\tau) 2\dot{\alpha}(t+\sigma)\dot{\beta}(t+\sigma) \cot \beta(t+\sigma)\, \mathrm{d}\sigma \\ \dot{\beta}(t) &= -\int_0^\infty \exp(-\sigma/\tau)\dot{\alpha}^2(t+\sigma) \sin \beta(t+\sigma) \cos \beta(t+\sigma)\, \mathrm{d}\sigma \end{aligned} \tag{14}$$

which can be put in the form

$$\begin{aligned} \dot{\alpha} &= \omega + T(2\dot{\alpha}\dot{\beta} \cot \beta) \\ \dot{\beta} &= -T(\dot{\alpha}^2 \sin \beta \cos \beta). \end{aligned} \tag{15}$$

After formally defining the operator T as

$$\begin{aligned} Tf(t) &= \int_0^\infty \exp(-\sigma/\tau) f(t+\sigma)\, \mathrm{d}\sigma = \sum_0^\infty \int_0^\infty \exp(-\sigma/\rho)(f^{(n)}(t)/n!)\sigma^n\, \mathrm{d}\sigma \\ &= \sum_0^\infty f^{(n)}(t)\tau^{n+1}. \end{aligned}$$

As τ is very small the approximation $Tf = \tau f$ is adequate and we can write equation (15) as

$$\begin{aligned} \dot{\alpha} &= \omega + 2\tau\dot{\alpha}\dot{\beta} \cot \beta \\ \dot{\beta} &= -\dot{\alpha}^2 \tau \sin \beta \cos \beta. \end{aligned} \tag{16}$$

For magnetic fields attainable in the laboratories the second term of the first equation can be neglected. For instance, if $B = 10^4$ G in the case of an electron $\dot{\alpha} \simeq 10^{11}\ \mathrm{s}^{-1}$, $\dot{\alpha}^2\tau \simeq 10^{-1}\ \mathrm{s}^{-1}$ and $\tau\dot{\alpha}\dot{\beta} \simeq 10^{-13}\ \mathrm{s}^{-1}$.

In that case the solution is, to an extremely good approximation,

$$\begin{aligned} \dot{\alpha} &= \omega \\ \tan \beta &= \tan \beta_0 \exp(-\omega^2 \tau t) \end{aligned} \tag{17}$$

which represents the spiralling of the spin towards $\beta = 0$ if $\beta_0 < \pi/2$ and towards $\beta = \pi$ if $\beta_0 > \pi/2$. Any unpolarised beam will evolve in such a way that, after a certain time, half of the spins will be parallel and half of them antiparallel to the magnetic field. If this field is inhomogeneous a transverse force will split the beam into two, producing two spots on a photographic plate. This would happen in a completely classical deterministic way. As is well known, the current interpretation of the Stern–Gerlach experiments uses a different kind of alignment based on the hypothesis of the reduction of the wave packet. It is nevertheless necessary to indicate that the effect that we are discussing is very weak because of the smallness of τ. For instance, β goes from 45° to 5° (or from 135° to 175°) in a time of 4·1 s in the case of an electron in a field of $B = 10^4$ G, and this time is proportional to B^{-2}.

It must be stressed that the effect is only produced if the particle is charged, because as we see in equation (8) it depends on the product $e\mu$. If the particle is neutral, as it is in the case of an atom, the correction to the Larmor precession is given by the third term in equation (8) which gives a smaller effect (the third term in the case of a hydrogen atom is about 10^{-10} times smaller than the second one for an electron in a field of $B = 10^4$ G) and always tends to make the spin parallel to B independently of β_0. For this reason the mechanism presented in this work cannot be applied to neutral atoms. Curiously enough, quantum mechanics predicts that the Stern–Gerlach effect cannot be observed with electrons (Baym 1969). Even with these restrictions, it is instructive to compare the classical and the quantum alignments. There are two important differences. First of all, in the classical mechanism there is an emission of magnetic dipole radiation with frequency $\omega = -eB/2mc$, which does not correspond to any quantum transition. In fact in the transition from spin up to spin down the emitted photons have a frequency 2ω. A second difference refers to the double Stern–Gerlach experiment. Let the $S_Z = \frac{1}{2}\hbar$ beam of a first Stern–Gerlach apparatus pass through a second one, forming an angle δ between the two magnetic fields. According to quantum mechanics there will be two spots with relative intensities $\cos^2(\delta/2)$ and $\sin^2(\delta/2)$. The mechanism presented in this work predicts only one spot corresponding to $+\frac{1}{2}$ if $\delta < \pi/2$ and to $-\frac{1}{2}$ if $\delta > \pi/2$.

4. Summary and conclusions

We have obtained radiative corrections to the Larmor frequency produced by the reaction to the emitted magnetic dipole radiation. In the case of charged particles, these corrections cause an alignment of the spin, such that it becomes parallel or antiparallel to the magnetic field if the initial angle between S and B is respectively smaller or greater than 90°. The effect, which would imply a classical explanation of the Stern–Gerlach experiment, is however too weak. The characteristic time is proportional to B^{-2} and is of the order of seconds for an electron in $B = 10^4$ G. It could be interesting to study the possibility of observing this effect either by detecting the emitted radiation or by a double Stern–Gerlach experiment. It could also prove useful to consider in detail the case of an atom, to which the present mechanism does not apply, to look for an alternative form of the same effect.

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